EXPLORING THE UNIVERSE ON THE BACK OF A GRAVITATIONAL WAVE *

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Abstract

The indirect interaction between cosmological gravitational waves (CGWs) and the surrounding matter content is considered, along with the evolution of the Universe from the inflationary epoch to the matter dominated era. Focusing on the power spectrum of the relic gravitational radiation, we arrive at a simple formula which relates the spectral index with the evolutionary period at which a CGW enters the horizon.

Key-words: Relic gravitational waves - inflation - evolution of the Universe

1. Introduction

The so-called cosmological gravitational waves (CGWs) represent small-scale perturbations to the Universal metric tensor (Weinberg 1972). Since gravity is the weakest of the four known forces, these metric corrections decouple from the rest of the Universe at very early times, presumably at the Planck epoch (Maggiore 2000). Their subsequent propagation is governed by the spacetime curvature (Misner et al 1973), encapsulating in the field equations the inherent coupling between relic GWs and the Universal matter content; the latter being responsible for the background gravitational field (Grishchuk and Polnarev 1980).

A GW background of cosmological origin is expected to be isotropic, stationary and unpolarized (Allen 1997). Therefore, its main property will be its frequency spectrum. Along with the propagation of relic GWs, the Universe experiences a number of (phase) transitions, mostly due to non-gravitational physics. The question

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that arises now is, if there are any imprints (of the various modifications in the spacetime dynamics) left on the spectrum of the CGWs during their journey from the inflationary epoch to the matter-dominated era. Provided that these imprints can be detected at the present epoch, CGWs would be a powerful tool to study Universal evolution.

The intensity of a GW background is characterized by the dimensionless quantity (Carr 1980)

$$\Omega_{gw} = \frac{1}{\rho_c}\frac{d\rho_{gw}}{d(lnk)} = \frac{1}{\rho_c}k\frac{d}{dk}\left[\frac{1}{2G}\int_0^\infty k^4 \left|\alpha(k,t)\right|^2 dk\right] \tag{1}$$

where, ρ_{gw} is the energy-density of the GW background, spatially averaged over several wavelengths, $\alpha(k,t)$ is the corresponding time-dependent amplitude, k is the coordinate wave-number and ρ_c is the present value of the critical energy-density for closing the Universe. In order to understand the effect of the GW background on a detector, we need to think in terms of amplitudes and therefore, it is convenient to express the *logarithmic spectrum* (1) in the form

$$\frac{d\rho_{gw}}{d(lnk)} = \frac{1}{2G}k^2\,\delta_h^2(k) \tag{2}$$

where,

$$\delta_h^2(k) = k^3\left|\alpha(k,t)\right|^2 \tag{3}$$

is the *power spectrum* (Mukhanov et al 1992), also referred to as the *characteristic amplitude* $h_c^2(k)$ (Maggiore 2000), which is a dimensionless quantity representing a characteristic value of the amplitude per unit of a logarithmic frequency interval. Clearly, in order to determine the spectrum of relic gravitational radiation one should first determine $\alpha(k,t)$ in curved spacetime. The safest way to do so, is to evaluate a family of solutions to the corresponding equation of propagation.

In the present article we consider a plane polarized gravitational wave (in the transverse-traceless gauge) propagating in a spatially flat Friedmann - Robertson - Walker (FRW) model. This model appears to interpret adequately the observational data related to the known thermal history of the Universe and therefore, it seems to be the most appropriate candidate for the curved background needed for this study. Accordingly, we arrive at a simple formula for the power spectrum of relic gravitational radiation, which relates the spectral index with the evolutionary period at which a CGW has entered the horizon.

2. Gravitational waves in curved spacetime

The far-field propagation (i.e. away from the source) of a weak CGW ($|h_{\mu\nu}| \ll 1$) in a curved, non-vacuum spacetime, is determined by the differential equations (Misner et al 1973)

$$h_{\mu\nu;\alpha}^{;\alpha} - 2\mathcal{R}_{\alpha\mu\nu\beta}h^{\alpha\beta} = 0 \tag{4}$$

under the gauge choice

$$(h^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} h)_{;\beta} = 0 \tag{5}$$

which brings the linearized Einstein equations into the form (4). In Eqs (4) and (5), Greek indices refer to the four-dimensional spacetime, $\mathcal{R}_{\alpha\mu\nu\beta}$ is the Riemann curvature tensor of the background metric, h is the trace of $h_{\mu\nu}$ and the semicolon denotes covariant derivative. A linearly polarized plane GW propagating in a spatially flat FRW cosmological model, is defined by the expression (Allen 1997)

$$ds^2 = c^2 dt^2 - R^2(t)(\delta_{ik} + h_{ik}) dx^i dx^k \tag{6}$$

where, Latin indices refer to the three-dimensional spatial section, δ_{ik} is the Kronecker symbol and the dimensionless scale factor $R(t)$ is a solution to the Friedmann equations, with mater-content in the form of a perfect fluid. Introducing the so-called *conformal time* coordinate, as

$$\eta = \int \frac{dt}{R(t)} \quad , \qquad 0 < \eta < \infty \tag{7}$$

the CGW equation of propagation reads (Grishchuk 1975)

$$h''_{ik} + 2\frac{R'}{R} h'_{ik} + \delta^{lm} h_{ik,lm} = 0 \tag{8}$$

where, a prime denotes differentiation with respect to η and the comma denotes spatial derivative. To decompose Eq (8), we represent the metric corrections h_{ik} in the form

$$h_{ik}(\eta, x^j) = \alpha(k, \eta)\, \varepsilon_{ik}\, e^{\imath k_j x^j} = \frac{h(\eta)}{R(\eta)}\, \alpha\, \varepsilon_{ik}\, e^{\imath k_j x^j} \tag{9}$$

where, $h(\eta)$ is the time-dependent part of the modes, k_j is the co-moving wave-vector, α is the dimensionless amplitude of the CGW and ε_{ik} is the corresponding polarization tensor. Combination of Eqs (8) and (9) results in a differential equation for the evolution of the time-dependent part of the modes

$$h'' + (k^2 c^2 - \frac{R''}{R}) h = 0 \tag{10}$$

To solve Eq (10), one needs an evolution formula for the cosmological model under consideration. In terms of the conformal time, the spatially flat FRW model considered, is a solution to the Friedmann equation

$$(\frac{R'}{R^2})^2 = \frac{8\pi G}{3}\, \rho(\eta) \tag{11}$$

with matter-content in the form of a perfect fluid, $T_{\mu\nu} = diag(\rho c^2, -p, -p, -p)$, which obeys the conservation law

$$\rho' + 3\frac{R'}{R}\, (\rho + \frac{1}{c^2}\, p) = 0 \tag{12}$$

and the equation of state

$$p = (\frac{m}{3} - 1)\ \rho c^2 \tag{13}$$

where, $\rho(\eta)$ and $p(\eta)$ represent the matter density and the pressure, respectively. The linear equation of state (13) covers most of the matter-components considered to drive the evolution of the Universe, such as quantum vacuum ($m = 0$), gas of strings ($m = 2$), dust ($m = 3$), radiation ($m = 4$) and Zel'dovich ultra-stiff matter ($m = 6$). For each component, the continuity equation (12) yields

$$\rho = \frac{M_m}{R^m} \tag{14}$$

where, M_m is an integration constant, representing the *total density* attributed to the $m-$th component. On the other hand, a mixture of these components obey (Bleyer et al 1991)

$$\rho = \sum_m \frac{M_m}{R^m} \tag{15}$$

where, now, Eq (12) holds for every matter-component separately. In this case, M_m represent the amount of the $m-$th component in the mixture and may vary asymptotically. Accordingly, for $M_m \to \infty$, the $m-$th component dominates over the others, while, for $M_m \to 0$, the $m-$th component is negligible in the composition of the mixture.

3. The power spectrum of relic gravitational radiation

In the case of an one-component fluid, the Friedmann equation reads

$$R^{\frac{m}{2}-2}\ R' = (\frac{8\pi G}{3} M_m)^{1/2} \tag{16}$$

admitting two families of exact solutions:

- For the gas of strings ($m = 2$)

$$R(\eta) \sim \exp(\sqrt{\frac{8\pi G}{3} M_2}\ \eta) \tag{17}$$

- For all the other types of matter-content ($m \neq 2$)

$$R(\tau) = (\frac{\tau}{\tau_m})^{\frac{2}{m-2}} \tag{18}$$

  where, the time-parameter τ is linearly related to the corresponding conformal one, by $\tau = \frac{m-2}{2}\eta$ and we have set $\tau_m = (\frac{8\pi G}{3} M_m)^{-1/2}$. Notice that, for $m = 0$ (De Sitter inflation) and $0 < \eta < \infty$, one obtains $-\infty < \tau < 0$.

In accordance, we obtain two families of solutions regarding the temporal evolution of a CGW in curved spacetime:

- Inserting Eq (17) into Eq (10) we obtain the temporal evolution of a CGW in a string-dominated Universe

$$h'' + (k^2c^2 - \frac{8\pi G}{3} M_2)h = 0 \quad \Rightarrow \quad h_2(k,\eta) = \sqrt{\eta}\, H_{1/2}^{(1,2)}(\omega\eta) \tag{19}$$

  where, $H_{1/2}^{(1,2)}$ denotes a linear combination of the Hankel's functions of the first and the second kind, of order 1/2 and

$$\omega^2 = k^2c^2 - \frac{8\pi G}{3}\, M_2 \tag{20}$$

  is the constant frequency of the CGW.

- Similarly, inserting Eq (18) into Eq (10) one obtains the temporal evolution of a CGW within the context of the inflationary and/or the standard model scenario

$$h'' + (k_*^2c^2 - 2[\frac{4-m}{(m-2)^2}]\,\frac{1}{\tau^2})h = 0 \;\Rightarrow\; h_m(k_*,\tau) = \sqrt{\tau}\, H_{|\nu|}^{(1,2)}(k_*c\tau) \tag{21}$$

  where, in this case, a prime denotes derivative with respect to τ, $k_* = \frac{2}{m-2}k$, so that $k_*c\tau = kc\eta$ and the Hankel's functions order is

$$\nu = \frac{1}{2}(\frac{m-6}{m-2}) \tag{22}$$

  (e.g. see Gradshteyn and Ryzhik 1965, Eq 8.491.5, p. 971). Therefore, different evolutionary periods admit different Hankel's functions.

With these solutions at hand, we may now examine the resulting power spectrum which can reveal a great deal of information on the physical conditions of the Universe at the time the CGWs have entered the horizon, i.e when the *physical wavelength* (λ_{ph}) of the wave becomes of the order of the Universal circumference

$$\lambda_{ph} \leq 2\pi\ell_H \Rightarrow \frac{2\pi}{k}R(\tau) \leq 2\pi\frac{c}{H} \Rightarrow k_*c\tau \geq 1 \Rightarrow kc\eta \geq 1 \tag{23}$$

Since a string-dominated Universe does not seem likely (Hindmarsh and Kibble 1995), in what follows, we confine ourselves within the inflationary and/or the standard model scenario, in which, the power spectrum is written in the form

$$\delta_h \sim k^{3/2}\,\tau^{\nu}\,\left|H_{|\nu|}^{(1,2)}(k_*c\tau)\right| \tag{24}$$

We define the *spectral index* as

$$n = \frac{k}{\delta_h(k)}\,\frac{d\delta_h(k)}{dk} \tag{25}$$

and taking into account the asymptotic properties of the Hankel functions (Lebedev 1972), we consider the following cases:

- A CGW is *well-outside* the horizon: In this case, we have $\lambda_{ph} \gg \ell_H$ and $k_* c\tau \to 0$. Accordingly, the power spectrum results in

$$\delta_h(k) \sim k^{\frac{3}{2}-|\nu|}\, \tau^{\nu-|\nu|} \tag{26}$$

and the spectral index is given by

$$n = \frac{3}{2} - |\nu| \tag{27}$$

In the case of inflationary expansion ($m = 0$), as well as during the matter-dominated epoch ($m = 3$) we obtain $n = 0$, i.e. the CGW spectrum is *flat*, as it has already been predicted by many authors (e.g. see Mukhanov et al 1992, Allen 1997). On the other hand in the radiation-dominated epoch ($m = 4$) one is left with the Harrison - Zel'dovich slope, where $n = 1$ and $\delta_h \sim k$.

- A CGW is *well-inside* the horizon: In this case, $\lambda_{ph} \ll \ell_H$ and $k_* c\tau \to \infty$. Accordingly, the power spectrum reads

$$\delta_h \sim k\, \tau^{\nu-\frac{1}{2}} \tag{28}$$

and the spectral index is constant ($n = 1$) for every m, yielding the Harrison - Zel'dovich slope (a not unexpected result) (e.g. see Mukhanov et al 1992).

- Finally, when the CGW enters the horizon one admits $\lambda_{ph} \simeq \ell_H \Rightarrow k_* c\tau \simeq 1$, thus obtaining

$$\delta_h \sim k^{\frac{3}{2}-\nu} \left| H^{(1,2)}_{|\nu|}(1) \right| \tag{29}$$

In this case, the Hankel's function is independent of k and, therefore, the spectral index reads

$$n = \frac{m}{m-2} \tag{30}$$

Eq (30) decomposes to

  - $n = 0$ (a flat spectrum) in the inflationary regime ($m = 0$)
  - $n = 2$ ($\delta_h \sim k^2$) in the radiation-dominated epoch ($m = 4$)
  - $n = 3$ ($\delta_h \sim k^3$) in the matter-dominated epoch ($m = 3$)

Therefore, knowledge of the spectral index allows us to determine the value of m which, in turn, determines the form of the matter-energy content of the Universe at the time the CGW has entered the horizon

$$m = \frac{2n}{n-1} \tag{31}$$

In this sense, a possible detection of CGWs can result in a powerful tool for exploring the Universe, even more powerful than the CMRB.

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