INTRODUCING QUADRATIC GRAVITY *

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Abstract

A novel approach in the semiclassical interaction of gravity with a quantum scalar field is considered, to guarantee the renormalizability of the energy-momentum tensor in a multi-dimensional curved spacetime. According to it, a self-consistent coupling between the square curvature term $\mathcal{R}^2$ and the quantum field is introduced. The subsequent interaction discards any higher-order derivative terms from the gravitational field equations, but, in the expense, it introduces a *geometric source* term in the wave equation for the quantum field. Unlike the conformal coupling case, this term does not represent an additional "mass" and, therefore, the quantum field interacts with gravity in a generic way and not only through its mass (or energy) content.

Key-words: Quantum gravity - semiclassical theory - gravity quintessence

1. Introduction

In the last few decades there has been a remarkable progress in understanding the quantum structure of the non-gravitational fundamental interactions (Nanopoulos 1997). On the other hand, so far, there is no quantum framework consistent enough to describe gravity itself (Padmanabhan 1989), leaving string theory as the most successful attempt towards this direction (Green et al 1987, Polchinsky 1998, Schwarz 1999). Within the context of General Relativity (GR), one usually resorts to perturbations' approach, where string theory predicts corrections to the Einstein equations. Those corrections originate from higher-order curvature terms arising in the string action, but their exact form is not yet being fully explored (Polchinsky 1998).

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A self-consistent mathematical background for *higher-order gravity theories* was formulated by Lovelock (1971). According to it, the most general gravitational Lagrangian reads

$$\mathcal{L} = \sqrt{-g} \sum_{m=0}^{n/2} \lambda_m \mathcal{L}^{(m)} \tag{1}$$

where λ_m are constant coefficients, n denotes the spacetime dimensions, g is the determinant of the metric tensor and $\mathcal{L}^{(m)}$ are functions of the Riemann curvature tensor $\mathcal{R}_{ijkl}$ and its contractions $\mathcal{R}_{ij}$ and $\mathcal{R}$, of the form

$$\mathcal{L}^{(m)} = \frac{1}{2^m} \delta^{j_1 ... j_{2m}}_{i_1 ... i_{2m}} \mathcal{R}^{i_1 i_2}_{j_1 j_2} ... \mathcal{R}^{i_{2m-1} i_{2m}}_{j_{2m-1} j_{2m}} \tag{2}$$

where Latin indices refer to the n-dimensional spacetime and $\delta^{j_1 ... j_{2m}}_{i_1 ... i_{2m}}$ is the generalized Kronecker symbol. In Eq (2), $\mathcal{L}^{(1)} = \frac{1}{2}\mathcal{R}$ is the Einstein-Hilbert (EH) Lagrangian, while $\mathcal{L}^{(2)}$ is a particular combination of quadratic terms, known as the Gauss-Bonnett (GB) combination, since in four dimensions it satisfies the functional relation

$$\frac{\delta}{\delta g^{\mu\nu}} \int \sqrt{-g} \left(\mathcal{R}^2 - 4 \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\kappa\lambda} \mathcal{R}^{\mu\nu\kappa\lambda} \right) d^4x = 0 \tag{3}$$

corresponding to the GB theorem (Kobayashi and Nomizu 1969). In Eq (3), Greek indices refer to four-dimensional coordinates. Introducing the GB term into the gravitational Lagrangian will not affect the four-dimensional field equations at all. However, within the context of the perturbations' approach mentioned above, the most important contribution comes from the GB term (Mignemi and Stewart 1993). As in four dimensions it is a total divergence, to render this term dynamical, one has to consider a higher-dimensional background or to couple it to a scalar field.

The idea of a multi-dimensional spacetime has received much attention as a candidate for the unification of all fundamental interactions, including gravity, in the framework of *super-gravity* and *super-strings* (Applequist et al 1987, Green et al 1987). In most higher-dimensional theories of gravity, the extra dimensions are assumed to form, at the present epoch, a compact manifold (*internal space*) of very small size compared to that of the three-dimensional visible space (*external space*) and therefore they are unobservable at the energies currently available (Green et al 1987). This so-called *compactification* of the extra dimensions may be achieved, in a natural way, by adding a square-curvature term ($\mathcal{R}_{ijkl}\mathcal{R}^{ijkl}$) in the EH action of the gravitational field (Müller-Hoissen 1988). In this way, the higher-dimensional theories are closely related to those of non-linear Lagrangians and their combination probably yields a natural generalization of GR.

In the present paper, we explore this generalization, in view of the renormalizable energy-momentum tensor which acts as the source of gravity in the (semiclassical) interaction between the gravitational and a quantum matter field. In particular:

We discuss briefly how GR is modified by the introduction of the renormalizable energy-momentum tensor first recognized by Calan et al (1970), on the rhs of the field equations. Introducing an analogous method, we explore the corresponding implications as regards a multi-dimensional higher-order gravity theory. We find that, in this case, the action functional, describing the semi-classical interaction of a quantum scalar field with the classical gravitational one, is being further modified and its variation with respect to the quantum field results in an inhomogeneous Klein-Gordon equation, the source term of which is purely geometric ($\sim \mathcal{R}^2$).

2. A Quadratic Interaction

Conventional gravity in $n-$dimensions implies that the dynamical behavior of the gravitational field arises from an action principle involving the EH Lagrangian

$$\mathcal{L}_{EH} = \frac{1}{16\pi G_n} \mathcal{R} \tag{4}$$

where, $G_n = GV_{n-4}$ and V_{n-4} denotes the volume of the internal space, formed by some extra spacelike dimensions. In this framework, we consider the semi-classical interaction between the gravitational and a massive quantum scalar field $\Phi(t, \vec{x})$ to the lowest order in G_n. The quantization of the field $\Phi(t, \vec{x})$ is performed by imposing canonical commutation relations on a hypersurface $t =$ constant (Isham 1981)

$$\begin{aligned} [\Phi(t, \vec{x}) \ , \ \Phi(t, \vec{x}')] = 0 = [\pi(t, \vec{x}) \ , \ \pi(t, \vec{x}')] \\ [\Phi(t, \vec{x}) \ , \ \pi(t, \vec{x}')] = i\delta^{(n-1)} \left(\vec{x} - \vec{x}' \right) \end{aligned} \tag{5}$$

where, $\pi(t, \vec{x})$ is the momentum canonically conjugate to the field $\Phi(t, \vec{x})$. The equal-time commutation relations (5) guarantee the local character of the quantum field theory under consideration, thus attributing its time-evolution to the classical gravitational field equations (Birrell and Davies 1982).

In any local field theory, the corresponding energy-momentum tensor is a very important object. Knowledge of its matrix elements is necessary to describe scattering in a relatively-weak external gravitational field. Therefore, in any quantum process in curved spacetime, it is desirable for the corresponding energy-momentum tensor to be *renormalizable*; i.e. its matrix elements to be cut-off independent (Birrell and Davies 1982). In this context, it has been proved (Callan et al 1970) that the functional form of the renormalizable energy-momentum tensor involved in the semi-classical interaction between the gravitational and a quantum field in $n-$dimensions, should be

$$\Theta_{ik} = T_{ik} - \frac{1}{4} \frac{n-2}{n-1} \left[\Phi^2{}_{;ik} - g_{ik} \Box \Phi^2 \right] \tag{6}$$

where, the semicolon stands for covariant differentiation (∇_k), $\Box = g^{ik} \nabla_i \nabla_k$ is the d' Alembert operator and

$$T_{ik} = \Phi_{,i} \Phi_{,k} - g_{ik} \mathcal{L}_{mat} \tag{7}$$

is the *conventional* energy-momentum tensor of an (otherwise) free massive scalar field, with Lagrangian density of the form

$$\mathcal{L}_{mat} = \frac{1}{2} \left[g^{ik} \Phi_{,i} \Phi_{,k} - m^2 \Phi^2 \right] \tag{8}$$

It is worth noting that the tensor (6) defines the same $n-$momentum and Lorentz generators as the conventional energy-momentum tensor.

It has been shown (Callan et al 1970) that the energy-momentum tensor (6) can be obtained by an action principle, involving

$$S = \int \left[f(\Phi)\mathcal{R} + \mathcal{L}_{mat} \right] \sqrt{-g} d^n x \tag{9}$$

where, $f(\Phi)$ is an arbitrary, analytic function of $\Phi(t, \vec{x})$, the determination of which can be achieved by demanding that the rhs of the field equations resulting from Eq (9) is given by Eq (6). Accordingly,

$$\frac{\delta S}{\delta g^{ik}} = 0 \Rightarrow \mathcal{R}_{ik} - \frac{1}{2} g_{ik} \mathcal{R} = -8\pi G_n \Theta_{ik} = -\frac{1}{2f} \left(T_{ik} + 2f_{;\,ik} - 2g_{ik} \Box f \right) \tag{10}$$

To lowest order in G_n, one obtains (Callan et al 1970)

$$f(\Phi) = \frac{1}{16\pi G_n} - \frac{1}{8} \frac{n-2}{n-1} \Phi^2 \tag{11}$$

Therefore, in any *linear Lagrangian gravity theory*, the interaction between a quantum scalar field and the classical gravitational one is determined through Hamilton's principle involving the action scalar

$$S = \int \sqrt{-g} \left[\left(\frac{1}{16\pi G_n} - \frac{1}{8} \frac{n-2}{n-1} \Phi^2 \right) \mathcal{R} + \mathcal{L}_{mat} \right] d^n x \tag{12}$$

On the other hand, both super-string theories (Candelas et al 1985, Green et al 1987) and the one-loop approximation of quantum gravity (Kleidis and Papadopoulos 1998), suggest that the presence of quadratic terms in the gravitational action is *a priori* expected. Therefore, in connection to the semi-classical interaction previously stated, the question that arises now is, what the functional form of the corresponding *renormalizable* energy-momentum tensor might be, if the simplest quadratic curvature term, $\mathcal{R}^2$, is included in the description of the classical gravitational field. To answer this question, by analogy to Eq (9), we may consider the action principle

$$\frac{\delta}{\delta g^{ik}} \int \sqrt{-g} \left[f_1(\Phi)\mathcal{R} + \alpha f_2(\Phi)\mathcal{R}^2 + \mathcal{L}_{mat} \right] d^n x = 0 \tag{13}$$

where, both $f_1(\Phi)$ and $f_2(\Phi)$ are arbitrary, polynomial functions of Φ. Eq (13) yields

$$\mathcal{R}_{ik} - \frac{1}{2} g_{ik} \mathcal{R} = -\frac{1}{2F} \left[T_{ik} + 2F_{;\,ik} - 2g_{ik} \Box F + \alpha g_{ik} f_2(\Phi) \mathcal{R}^2 \right] \tag{14}$$

where, the function F stands for the combination

$$F = f_1(\Phi) + 2\alpha \mathcal{R} f_2(\Phi) \tag{15}$$

For $\alpha = 0$ and to the lowest order in G_n (but to every order in the coupling constants of the quantum field involved), we must have

$$\mathcal{R}_{ik} - \frac{1}{2} g_{ik} \mathcal{R} = -8\pi G_n \Theta_{ik} \tag{16}$$

where, Θ_{ik} [given by Eq (6)] is the renormalizable energy-momentum tensor first recognized by Calan et al (1970). In this respect, we obtain $f_1(\Phi) = f(\Phi)$, i.e. a function quadratic in Φ [see Eq (11)]. Furthermore, on dimensional grounds regarding Eq (14), we expect that

$$F \sim \Phi^2 \tag{17}$$

and, therefore, $\alpha \mathcal{R} f_2(\Phi) \sim \Phi^2$, as well. However, we already know that $\mathcal{R} \sim [\Phi]$, as indicated by Whitt (1984), something that leads to $f_2(\Phi) \sim \Phi$ [1] and in particular,

$$F(\Phi) = \frac{1}{16\pi G_n} - \frac{1}{8}\frac{n-2}{n-1}\left[\Phi^2\right] + 2\alpha\mathcal{R}\Phi \tag{18}$$

In Eq (18), the coupling parameter α encapsulates any arbitrary constant that may be introduced in the definition of $f_2(\Phi)$. Accordingly, the action describing the semi-classical interaction of a quantum scalar field with the classical gravitational one up to the second order in curvature tensor, is being further modified and is written in the form

$$S = \int \sqrt{-g}\left[\left(\frac{1}{16\pi G_n} - \frac{1}{2}\xi_n\Phi^2\right)\mathcal{R} + \alpha\mathcal{R}^2\Phi + \mathcal{L}_{mat}\right] d^n x \tag{19}$$

where

$$\xi_n = \frac{1}{4}\frac{n-2}{n-1} \tag{20}$$

is the so-called *conformal coupling* parameter (Birrell and Davies 1982). In this case, the associated gravitational field equations (14) result in

$$\mathcal{R}_{ik} - \frac{1}{2}g_{ik}\mathcal{R} = -8\pi G_n\left(\Theta_{ik} + \alpha S_{ik}\right) \tag{21}$$

where

$$S_{ik} = g_{ik}\,\mathcal{R}^2\Phi \tag{22}$$

The rhs of Eq (21) represents the *"new"* renormalizable energy-momentum tensor. Notice that, as long as $\alpha \neq 0$, this tensor contains the extra *"source"* term S_{ik}. In spite the presence of this term, the generalized energy-momentum tensor still remains renormalizable. This is due to the fact that, the set of the quantum operators $\{\Phi, \Phi^2, \Box\Phi\}$ is closed under renormalization, as it can be verified by straightforward power counting (see Callan et al 1970).

Eq (22) implies that the quadratic curvature term (i.e. pure global gravity) acts as a source of the quantum field Φ. Indeed, variation of Eq (19) with respect to $\Phi(t, \vec{x})$ leads to the following quantum field equation of propagation

$$\Box\Phi + m^2\Phi + \xi_n\mathcal{R}\Phi = \alpha\mathcal{R}^2 \tag{23}$$

that is, an inhomogeneous Klein-Gordon equation in curved spacetime. It is worth pointing out that, in Eq (19), the generalized coupling constant α remains dimensionless (and this is also the case for the corresponding action) only as long as

$$n = 6 \tag{24}$$

[1] In fact, $[\mathcal{R}] \sim [\Phi]^{\frac{4}{n-2}}$ and, therefore, $f_2 \sim \Phi^{2\frac{n-4}{n-2}}$. In order to render the coupling constant α dimensionless, one should consider $n = 6$. Hence, $f_2 \sim \Phi$ only in six dimensions.

thus indicating the appropriate spacetime dimensions for the semi-classical theory under consideration to hold, without introducing any additional arbitrary length scales.

Summarizing, a self-consistent coupling between the square curvature term $\mathcal{R}^2$ and the quantum field $\Phi(t, \vec{x})$ should be introduced in order to yield the "correct" renormalizable energy-momentum tensor in non-linear gravity theories. The subsequent *quadratic interaction* discards any higher-order derivative terms from the gravitational field equations, but it introduces a *geometric source* term in the wave equation for the quantum field. In this case, unlike the *conventional* conformal coupling ($\sim \mathcal{R}\Phi^2$), the quantum field interacts with gravity not only through its mass (or energy) content ($\sim \Phi^2$), but, also, in a more generic way ($\mathcal{R}^2\Phi$).

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