I hope that you will agree that this beautiful model is a fitting climax to the projects on competing species.

4.17 A MODEL FOR THE SPREAD OF DISEASE

An infectious disease is spreading among a population of N individuals; all changes in the progress of the disease will be assumed to be continuous. The incubation period of the disease is neglected or assumed to be zero. Someone suffering from the disease may die or may be cured, in which case he or she becomes immune. The population contains the following groups:

I, *infective*. Those who have got it.
S, *susceptible*. Those who are in danger of getting it.
R, *removed*. Those who have died, those who are isolated for treatment, and those who have recovered and are immune.

If these symbols refer to actual numbers, then

$$N = I + S + R. \tag{4.17.1}$$

In practice, the symbols might represent units of thousands or even millions of individuals. (Scaling may be necessary on some computers to avoid overflow.) Note that in (4.17.1) we are assuming that there is no change in the population by emigration, by immigration, or by a nonbalanced natural birth and death rate.

We assume that the rate at which the disease spreads depends on the number of contacts between infected and susceptible individuals, so that it is proportional to the product SI. We also assume that the rate of removal is proportional to I. Then if the time t is the independent variable, the equations for the model are:

$$\left.\begin{aligned} \frac{dS}{dt} &= -aSI, \\ \frac{dI}{dt} &= -bI + aSI, \\ \frac{dR}{dt} &= bI. \end{aligned}\right] \tag{4.17.2}$$

Only the first two equations need to be worried about, since, once they are solved, R can be found from (4.17.1).

As with Volterra's predator–prey model, the time can be eliminated if the first equation is divided by the second; then the separable equation

$$\frac{dS}{dI} = \frac{aSI}{bI - aSI}$$

results, with solution

$$I = I_0 - (S - S_0) + \frac{b}{a} \ln (S/S_0). \tag{4.17.3}$$

Here, I_0 and S_0 are initial conditions. (Don't just sit there, check it for yourself!) Orbits in the S-I phase plane are shown in Figure 4.10 for the case $a = b = 1$; they move from right to left. The ratio b/a is important here. If S is initially less than b/a, then the disease immediately starts to die out. But if S is initially greater than b/a, there is an increase and we have an "epidemic."

A quantity of interest is the rate dR/dt at which infectives are removed. This rate might be the most visible sign of the disease so far as records are concerned; it might also be proportional to the death rate. Curves for $dR/dt = R'$ are plotted in Figure 4.11, with the same initial conditions as the cases plotted in Figure 4.10.

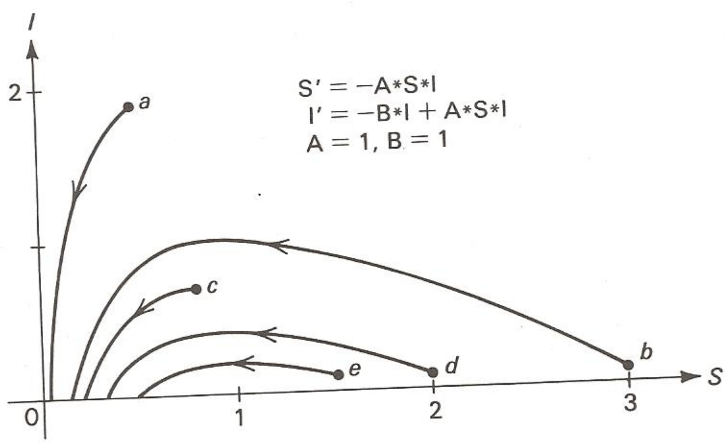

Figure 4.10 $S' = -SI$, $I' = -I + SI$. **Different orbits.**

Notice that with the unit of time free, we can set $a = 1$. Also, the "interesting" values of the parameters will depend on whether actual population figures are used, or, if they are scaled, how they are scaled. Be prepared to experiment. For your project, choose a ratio b/a. Start with a small number of infectives, and vary the initial value for S. Consider how things might change if the disease became more infectious (a increases) or the diagnosis more effective (b increases).

Discussion of these equations and many generalizations can be found in many texts. See particularly *The Mathematical Theory of Infectious Diseases and its Applications* by N. T. J. Bailey [3]. See also the text by Braun [5].

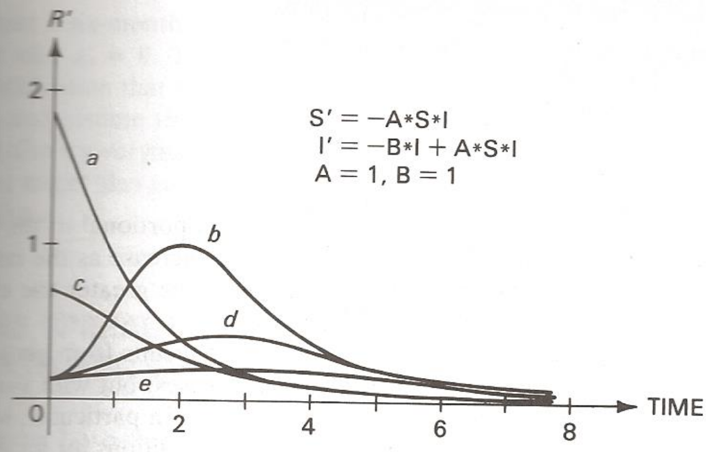

Figure 4.11 Same system as that for Figure 4.10. R' is the rate of removal.

4.18 CROSS-INFECTION BETWEEN TWO SPECIES

Here we consider a disease that can be transmitted between two distinct populations, but not between members of the same population. Let the two species have populations N_1 and N_2, and let S_i, I_i, and R_i be the numbers of susceptible, infected, and removed members, respectively, of the ith species, with $i = 1,2$. Then, as in the preceding project, we have the equations

$$\left.\begin{array}{ll} S_1' = -a_1S_1I_2, & S_2' = -a_2S_2I_1, \\ I_1' = -b_1I_1 + a_1S_1I_2, & I_2' = -b_2I_2 + a_2S_2I_1, \\ R_1' = b_1I_1, & R_2' = b_2I_2. \end{array}\right] \quad (4.18.1)$$

As before, there is no need to worry about the third equation in either set.

Experiment with these equations. Try setting all the parameters equal to one. You might start with the first population being entirely free of the disease, with just a few members of the second being infected. This model is rather general, so a lot is left to your initiative.

4.19 POSSIBLE EFFECTS OF VACCINATION

Vaccination will reduce the number of susceptibles, which means that we shall be modifying the first equation of (4.17.2). If the number vaccinated at any time is V, we need a model for dV/dt. Accordingly, consider

4.44 THE SIMPLE PENDULUM

We shall have a sequence of projects to do with pendulums, and it is fitting to start with the simplest. The model for the "simple pendulum" has a mass m connected to a fixed point O by a light rigid rod of length ℓ. The system is set up in a room where the constant acceleration due to gravity is g, and the rod and mass are free to move in a vertical plane through O. All resisting forces are neglected. If the rod makes the angle θ with the downward vertical, as shown in Figure 4.30, then the d.e. for the model is

$$\frac{d^2\theta}{dt^2} = \theta'' = -\frac{g}{\ell}\sin\theta. \qquad (4.44.1)$$

When θ is small, we have the well known approximation

$$\theta'' + \frac{g}{\ell}\theta = 0. \qquad (4.44.2)$$

This is the equation of a harmonic oscillator with frequency $\sqrt{g/\ell}$ and period $2\pi\sqrt{\ell/g}$. These are independent of the (small) amplitude of the swing. Also notice that the mass m does not appear in (4.44.1). If the frequency is called ω_0, then

$$\omega_0^2 = \frac{g}{\ell}, \qquad (4.44.3)$$

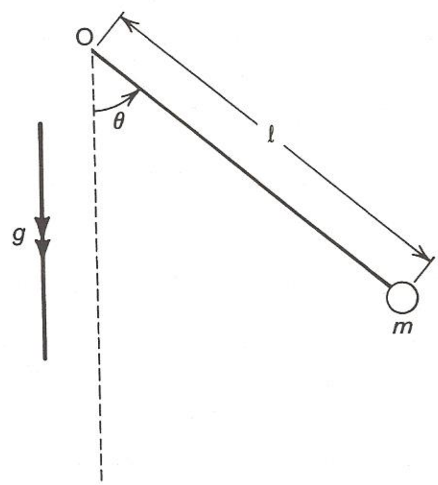

Figure 4.30 The simple pendulum.

and ω_0 is called the *natural frequency* of the system. In terms of ω_0, (4.44.1) can be written as

$$\theta'' + \omega_0^2 \sin\theta = 0. \tag{4.44.4}$$

The system has two equilibrium positions, one vertically down with $\theta = 0$ and stable, and the other vertically up with $\theta = \pi$ and unstable. The motion may consist of swings or "librations," or there may be "circulation."

The solutions are well described in the phase plane of θ and θ'. A phase plane diagram for several solutions is shown in Figure 4.31. Observe examples of the different sorts of motion. You are likely to see similar diagrams in many texts, but it is a good experience to generate at least one for yourself. Doing so is the first part of this project.

The period of oscillation for librational motion does in fact depend on the amplitude of the swing. Do some calculation to construct a table showing this dependence, and deduce from the table how large the amplitude must be for the approximation (4.44.2) to break down. Note that the period tends to infinity as the amplitude tends to π.

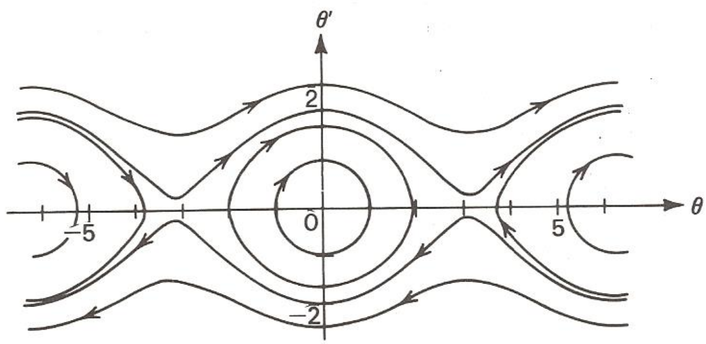

Figure 4.31 Phase diagram for $\theta'' + \sin\theta = 0$.

4.45 THE PENDULUM WITH LINEAR DAMPING

The equation for the model is

$$\theta'' + k\theta' + \omega_0^2 \sin\theta = 0, \tag{4.45.1}$$

where k is a positive constant. We assume that resisting forces (air resistance or friction at the point of suspension) make up a torque that is proportional to the rate of change of θ. As expected, all motion is damped: circulatory motion eventually becomes librational, and the amplitude of librational motion decreases to zero. Two phase diagrams are shown in Figure 4.32.

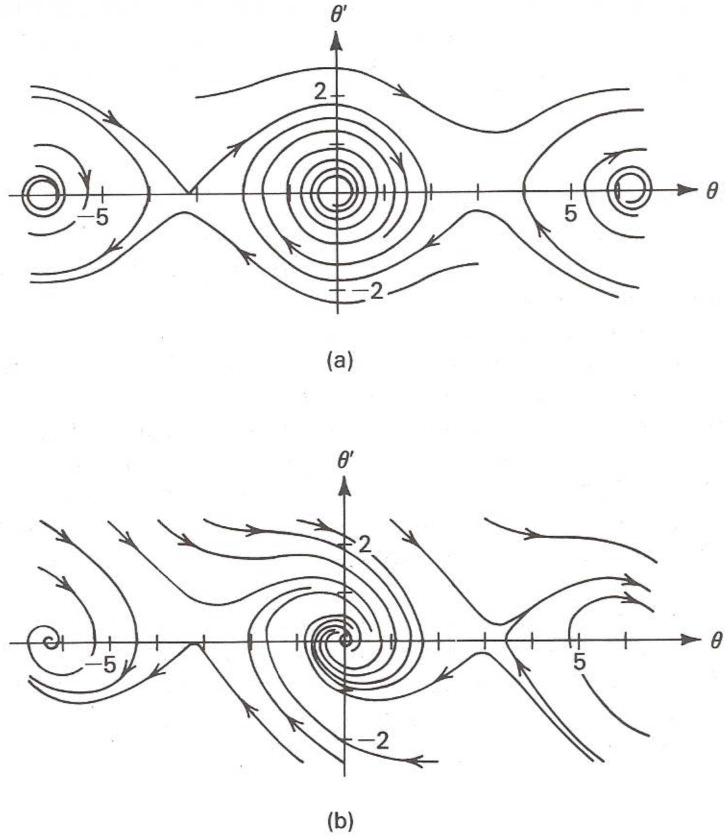

Figure 4.32 (a) Phase diagram for $\theta'' + 0.1\theta' + \sin\theta = 0$; (b) Phase diagram for $\theta'' + 0.5\theta' + \sin\theta = 0$.

As with the preceding project, make yourself familiar with the properties and possibilities of this model through your own computation. For a game, start the motion off with fast circulation, and see if you can guess in advance how many circuits will be described before librational motion takes over.

4.46 THE PENDULUM WITH DRY FRICTION

If the surface of contact between two solids is dry, then the force resisting the motion of one relative to the other may be described by "dry" or "Coulomb" friction. We shall look at two models for this phenomenon. In the first, the resisting force is constant in magnitude (with no dependence on the relative velocity between the surfaces). The resulting model is given by

$$\theta'' + k\,\mathrm{Sgn}(\theta') + \omega_0^2 \sin\theta = 0, \tag{4.46.1}$$

3. Motion with the pivot attached to the rim of a wheel which rotates at constant angular speed: $x_0 = A\cos(\omega t)$, $y_0 = A\sin(\omega t)$.

The mathematical discussion of this model is not easy, but numerical experimentation is most enjoyable. When planning which parameters and initial conditions to try, imagine that you are holding the pivot in your hand, and see if you can guess in advance what will happen.

This problem has attracted quite a lot of attention, and several papers have appeared in the literature—in particular, in the *American Journal of Physics* (a journal that you should read regularly as a source of readable ideas and good models). There are two articles by F. M. Phelps and J. H. Hunter ([43], [44]) that you should look at; in the second of these, a demonstration apparatus is described for showing possible stability in the laboratory. Of course, the computer is furnishing you with your own apparatus. Also, it is claimed that damping effects are not important for influencing the stability results. I suggest that when you have found a stable solution, you add some damping terms to the equations and check this assertion.

4.49 A PENDULUM WITH VARYING LENGTH

Here, the point of suspension might consist of a thin rod. The mass m is at the end of a light, inextensible string, and the other end of the string can be pulled over the rod. (See Figure 4.34.) Another way to think of this type of pendulum is to consider a child at the end of a swing. The swing has energy "pumped" into it through changes in the posture of the child, and these changes alter the distance of the center of mass of the system from the point of suspension O, or, essentially, the length of the pendulum.

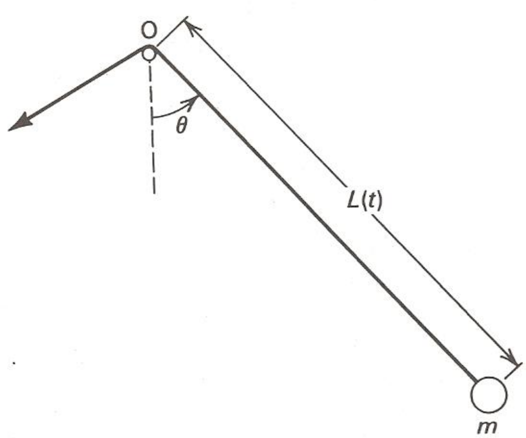

Figure 4.34

Let the variable length be $L(t)$; this will be a predetermined function in the model. If θ is defined, as before, as the angle between the pendulum and the vertical, then it can be shown that

$$L^2\theta'' + 2LL'\theta' + gL \sin \theta = 0. \tag{4.49.1}$$

Let

$$\theta = y_1 \quad \text{and} \quad \theta' = y_2. \tag{4.49.2}$$

Then (4.49.1) can be written as the system

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= -(g/L)\sin(y_1) - 2(L'/L)y_2. \end{aligned} \tag{4.49.3}$$

As always, you can add a damping term if you wish.

You can experiment with the system in several ways. Mostly, the variation in L will be oscillatory, so you might let

$$L(t) = L_0 + L_1 \sin(\omega t - \delta), \tag{4.49.4}$$

where L_1 is smaller, perhaps much smaller, than L_0. The natural frequency is ω_0, where $\omega_0^2 = g/L_0$. Experiment with values of ω close to the natural frequency. As ω approaches ω_0, you should see a dramatic resonance effect, with the amplitudes of the swings increasing. The phase angle δ controls where in the swing the maximum length occurs. What effect does this parameter have on the motion of the pendulum?

4.50 THE SPRING PENDULUM

In this model the mass m is at one end of a light spring; the other end is attached to the fixed point of suspension O. The spring remains straight, and the tension in it obeys Hooke's law; the spring constant is k. (See Figure 4.35.) Let the unstretched length of the spring be L_0, and its length at time t be $L(t)$. Then, resolving accelerations and forces along and perpendicular to the spring, we have

$$\begin{aligned} L'' - L\theta'^2 - g \cos \theta + \frac{k}{m}(L - L_0) &= 0, \\ L^2\theta'' + 2LL'\theta' + gL \sin \theta &= 0. \end{aligned} \tag{4.50.1}$$

There are two "natural frequencies" in the model: $\omega_1 = \sqrt{g/L_0}$ corresponds to the pendulum without the spring; and $\omega_2 = \sqrt{k/m}$ corresponds to the spring without the pendulum. The vibrations in L and θ can exchange energies in interesting ways, and this is especially noticeable if the ratio ω_1/ω_2 is close to a rational number like 1, 2, or ½. Then we can observe phenomena arising from